

# Chapter 1

## Lorenz-like chaotic attractors revised

Vítor Araújo and Maria José Pacifico

**Abstract** We describe some recent results on the dynamics of singular-hyperbolic (Lorenz-like) attractors  $\Lambda$  introduced in [25]: (1) there exists an invariant foliation whose leaves are forward contracted by the flow; (2) there exists a positive Lyapunov exponent at every orbit; (3) attractors in this class are expansive and so sensitive with respect to initial data; (4) they have zero volume if the flow is  $C^2$ , or else the flow is globally hyperbolic; (5) there is a unique physical measure whose support is the whole attractor and which is the equilibrium state with respect to the center-unstable Jacobian; (6) the hitting time associated to a geometric Lorenz attractor satisfies a logarithm law; (7) the rate of large deviations for the physical measure on the ergodic basin of a geometric Lorenz attractor is exponential.

**Key words:** singular-hyperbolic attractors, zero volume, physical measure, expansiveness, positive Lyapunov exponent, non-uniform expansion, hyperbolic times, large deviations, geometric Lorenz flows, hitting time, logarithm law, special flows.

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## 1.1 Introduction

In this note  $M$  is a compact boundaryless 3-manifold and  $\mathcal{X}^1(M)$  denotes the set of  $C^1$  vector fields on  $M$  endowed with the  $C^1$  topology. Moreover  $\text{Leb}$  denotes *volume* or *Lebesgue measure*: a normalized volume form given by some Riemannian metric on  $M$ . We also denote by  $\text{dist}$  the Riemannian distance on  $M$ .

The notion of singular hyperbolicity was introduced in [23, 25] where it was proved that any  $C^1$  robustly transitive set for a 3-flow is either a singular hyperbolic attractor or repeller.

A compact invariant set  $\Lambda$  of a 3-flow  $X \in \mathcal{X}^1(M)$  is an *attractor* if there exists a neighborhood  $U$  of  $\Lambda$  (its isolating neighborhood) such that

$$\Lambda = \bigcap_{t>0} X^t(U)$$

and there exists  $x \in \Lambda$  such that  $X(x) \neq 0$  and whose positive orbit  $\{X^t(x) : t > 0\}$  is dense in  $\Lambda$ .

We say that a compact invariant subset is *singular hyperbolic* if all the singularities in  $\Lambda$  are hyperbolic, and the tangent bundle  $T\Lambda$  decomposes in two complementary  $DX^t$ -invariant bundles  $E^s \oplus E^{cu}$ , where:  $E^s$  is one-dimensional and uniformly contracted by  $DX^t$ ;  $E^{cu}$  is bidimensional, contains the flow direction,  $DX^t$  expands area along  $E^{cu}$  and  $DX^t|_{E^{cu}}$  dominates  $DX^t|_{E^s}$  (i.e. any eventual contraction in  $E^s$  is stronger than any possible contraction in  $E^{cu}$ ), for all  $t > 0$ .

We note that the presence of an equilibrium together with regular orbits accumulating on it prevents any invariant set from being uniformly hyperbolic, see e.g. [12]. Indeed, in our 3-dimensional setting a compact invariant subset  $\Lambda$  is uniformly hyperbolic if the tangent bundle  $T\Lambda$  decomposes in *three* complementary  $DX^t$ -invariant bundles  $E^s \oplus E^X \oplus E^u$ , each one-dimensional,  $E^X$  is the flow direction,  $E^s$  is uniformly contracted and  $E^u$  uniformly expanded by  $DX^t$ ,  $t > 0$ . This implies the continuity of the splitting and the presence of a non-isolated equilibrium point in  $\Lambda$  leads to a discontinuity in the splitting dimensions.

In the study of the asymptotic behavior of orbits of a flow  $X \in \mathcal{X}^1(M)$ , a fundamental problem is to understand how the behavior of the tangent map  $DX$  determines the dynamics of the flow  $X_t$ . The main achievement along this line is the uniform hyperbolic theory: we have a complete description of the dynamics assuming that the tangent map has a uniformly hyperbolic structure since [12].

In the same vein, under the assumption of singular hyperbolicity, one can show that at each point there exists a strong stable manifold and that the whole set is foliated by leaves that are contracted by forward iteration. In particular this shows that any robust transitive attractor with singularities displays similar properties to those of the geometrical Lorenz model. It is also possible to show the existence of local central manifolds tangent to the central unstable direction. Although these central manifolds do not behave as unstable ones, in the sense that points on them are not necessarily asymptotic in the past, the expansion of volume along the central unstable two-dimensional direction enables us to deduce some remarkable properties.

We shall list some of these properties that give us a nice description of the dynamics of a singular hyperbolic attractor.

## 1.2 The geometric Lorenz attractor

Here we briefly recall the construction of the geometric Lorenz attractor [1, 15], that is the more representative example of a singular-hyperbolic attractor.

In 1963 the meteorologist Edward Lorenz published in the Journal of Atmospheric Sciences [26] an example of a parametrized polynomial system of differential equations

$$\begin{aligned} \dot{x} &= a(y - x) & a &= 10 \\ \dot{y} &= rx - y - xz & r &= 28 \\ \dot{z} &= xy - bz & b &= 8/3 \end{aligned} \tag{1.1}$$

as a very simplified model for thermal fluid convection, motivated by an attempt to understand the foundations of weather forecast.

The origin  $\sigma = (0, 0, 0)$  is an equilibrium of saddle type for the vector field defined by equations (1.1) with real eigenvalues  $\lambda_i$ ,  $i \leq 3$  satisfying

$$\lambda_2 < \lambda_3 < 0 < -\lambda_3 < \lambda_1. \tag{1.2}$$

(in this case  $\lambda_1 \approx 11.83$ ,  $\lambda_2 \approx -22.83$ ,  $\lambda_3 = -8/3$ ).

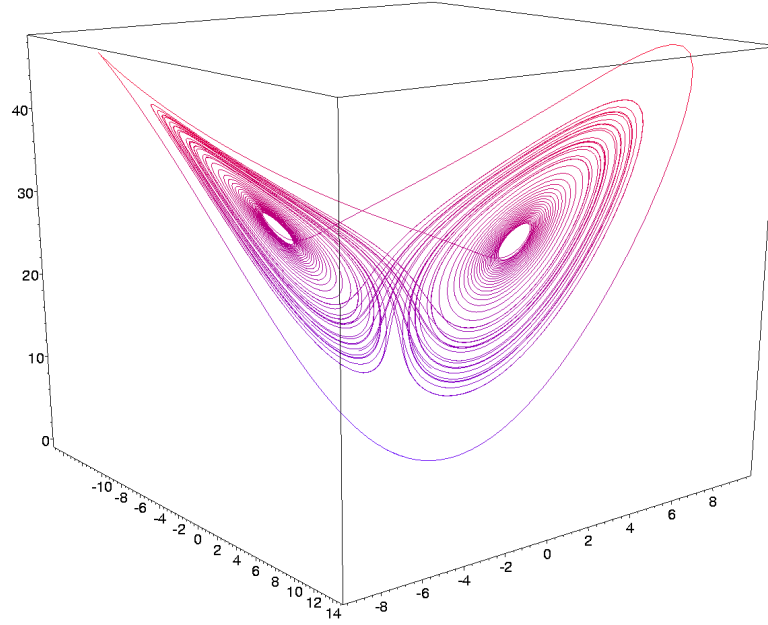
Numerical simulations performed by Lorenz for an open neighborhood of the chosen parameters suggested that almost all points in phase space tend to a *chaotic attractor*, whose well known picture is presented in Figure 1.1. The *chaotic feature* is the fact that trajectories converging to the attractor are *sensitive with respect to initial data*: trajectories of any two nearby points are driven apart under time evolution.

Lorenz's equations proved to be very resistant to rigorous mathematical analysis, and also presented serious difficulties to rigorous numerical study. Indeed, these two main difficulties are:

*conceptual*: the presence of an equilibrium point at the origin accumulated by regular orbits of the flow prevents this attractor from being hyperbolic [7],

*numerical*: the presence of an equilibrium point at the origin, implying that solutions slow down as they pass near the origin, which means unbounded return times and, thus, unbounded integration errors.

Moreover the attractor is *robust*, that is, the features of the limit set persist for all nearby vector fields. More precisely, if  $U$  is an isolating neighborhood of the attractor  $\Lambda$  for a vector field  $X$ , then  $\Lambda$  is *robustly transitive* if, for all vector fields  $Y$  which are  $C^1$  close to  $X$ , the corresponding  $Y$ -invariant set



**Fig. 1.1** A view of the Lorenz attractor calculated numerically

$$\Lambda_Y(U) = \bigcup_{t>0} Y^t(U)$$

also admits a dense positive  $Y$ -orbit. We remark that *the persistence of transitivity*, that is, the fact that, for all nearby vector fields, the corresponding limit set is transitive, *implies a dynamical characterization of the attractor*, as we shall see.

These difficulties led, in the seventies, to the construction of a geometric flow presenting a similar behavior as the one generated by equations (1.1). Nowadays this model is known as *geometric Lorenz flow*. Next we briefly describe this construction, see [1, 15] for full details.

We start by observing that under some non-resonance conditions, by the results of Sternberg [30], in a neighborhood of the origin, which we assume to contain the cube  $[-1, 1]^3 \subset \mathbb{R}^3$ , the Lorenz equations are equivalent to the linear system  $(\dot{x}, \dot{y}, \dot{z}) = (\lambda_1 x, \lambda_2 y, \lambda_3 z)$  through smooth conjugation, thus

$$X^t(x_0, y_0, z_0) = (x_0 e^{\lambda_1 t}, y_0 e^{\lambda_2 t}, z_0 e^{\lambda_3 t}), \quad (1.3)$$

where  $\lambda_1 \approx 11.83$ ,  $\lambda_2 \approx -22.83$ ,  $\lambda_3 = -8/3$  and  $(x_0, y_0, z_0) \in \mathbb{R}^3$  is an arbitrary initial point near  $(0, 0, 0)$ .

Consider  $S = \{(x, y, 1) : |x| \leq 1/2, |y| \leq 1/2\}$  and

$$\begin{aligned} S^- &= \{(x, y, 1) \in S : x < 0\}, & S^+ &= \{(x, y, 1) \in S : x > 0\} \quad \text{and} \\ S^* &= S^- \cup S^+ = S \setminus \ell, & \text{where } \ell &= \{(x, y, 1) \in S : x = 0\}. \end{aligned}$$

Assume that  $S$  is a global transverse section to the flow so that every trajectory eventually crosses  $S$  in the direction of the negative  $z$  axis.

Consider also  $\Sigma = \{(x, y, z) : |x| = 1\} = \Sigma^- \cup \Sigma^+$  with  $\Sigma^\pm = \{(x, y, z) : x = \pm 1\}$ .

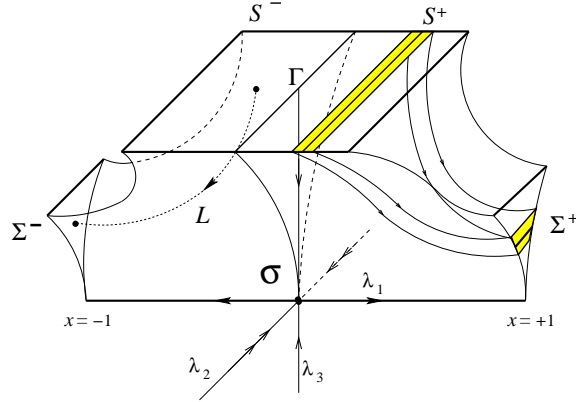
For each  $(x_0, y_0, 1) \in S^*$  the time  $\tau$  such that  $X^\tau(x_0, y_0, 1) \in \Sigma$  is given by

$$\tau(x_0) = -\frac{1}{\lambda_1} \log |x_0|,$$

which depends on  $x_0 \in S^*$  only and is such that  $\tau(x_0) \rightarrow +\infty$  when  $x_0 \rightarrow 0$ . This is one of the reasons many standard numerical algorithms were unsuited to tackle the Lorenz system of equations. Hence we get (where  $\text{sgn}(x) = x/|x|$  for  $x \neq 0$ )

$$X^\tau(x_0, y_0, 1) = (\text{sgn}(x_0), y_0 e^{\lambda_2 \tau}, e^{\lambda_3 \tau}) = (\text{sgn}(x_0), y_0 |x_0|^{-\frac{\lambda_2}{\lambda_1}}, |x_0|^{-\frac{\lambda_3}{\lambda_1}}). \quad (1.4)$$

Since  $0 < -\lambda_3 < \lambda_1 < -\lambda_2$ , we have  $0 < \alpha = -\frac{\lambda_3}{\lambda_1} < 1 < \beta = -\frac{\lambda_2}{\lambda_1}$ . Let  $L : S^* \rightarrow \Sigma$  be such that  $L(x, y) = (y|x|^\beta, |x|^\alpha)$  with the convention that  $L(x, y) \in \Sigma^+$  if  $x > 0$  and  $L(x, y) \in \Sigma^-$  if  $x < 0$ . It is easy to see that  $L(S^\pm)$  has the shape of a triangle without



**Fig. 1.2** Behavior near the origin.

the vertex  $(\pm 1, 0, 0)$ . In fact the vertex  $(\pm 1, 0, 0)$  are cusp points at the boundary of each of these sets. The fact that  $0 < \alpha < 1 < \beta$  together with equation (1.4) imply that  $L(\Sigma^\pm)$  are uniformly compressed in the  $y$ -direction.

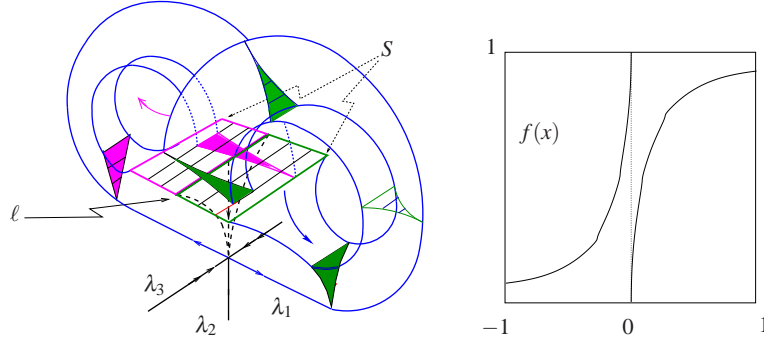
From now on we denote by  $\Sigma^\pm$  the closure of  $L(S^\pm)$ . Clearly each line segment  $S^* \cap \{x = x_0\}$  is taken to another line segment  $\Sigma \cap \{z = z_0\}$  as sketched in Figure 1.2.

The sets  $\Sigma^\pm$  should return to the cross section  $S$  through a composition of a translation  $T$ , an expansion  $E$  only along the  $x$ -direction and a rotation  $R$  around  $W^s(\sigma_1)$  and  $W^s(\sigma_2)$ , where  $\sigma_i$  are saddle-type singularities of  $X^t$  that are outside the cube  $[-1, 1]^3$ , see [7]. We assume that this composition takes line segments  $\Sigma \cap \{z =$

$z_0\}$  into line segments  $S \cap \{x = x_1\}$  as sketched in Figure 1.2. The composition  $T \circ E \circ R$  of linear maps describes a vector field  $V$  in a region outside  $[-1, 1]^3$ . The geometric Lorenz flow  $X^t$  is then defined in the following way: for each  $t \in \mathbb{R}$  and each point  $x \in S$ , the orbit  $X^t(x)$  will start following the linear field until  $\tilde{\Sigma}^\pm$  and then it will follow  $V$  coming back to  $S$  and so on. Let us write  $\mathcal{B} = \{X^t(x), x \in S, t \in \mathbb{R}^+\}$  the set where this flow acts. The geometric Lorenz flow is then the pair  $(\mathcal{B}, X^t)$  defined in this way. The set

$$\Lambda = \cap_{t \geq 0} X^t(S)$$

is the *geometric Lorenz attractor*.



**Fig. 1.3** The global cross-section for the geometric Lorenz flow and the associated 1d quotient map, the Lorenz transformation.

We remark that the existence of a chaotic attractor for the original Lorenz system was established by Tucker with the help of a computer aided proof (see [31]).

The combined effects of  $T \circ E \circ R$  and the linear flow given by equation (1.4) on lines implies that the foliation  $\mathcal{F}^s$  of  $S$  given by the lines  $S \cap \{x = x_0\}$  is invariant under the first return map  $F : S \rightarrow S$ . In another words, we have

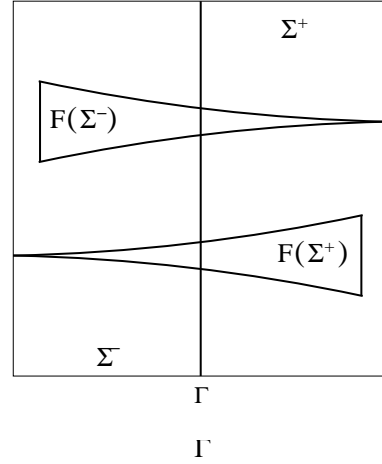
( $\star$ ) for any given leaf  $\gamma$  of  $\mathcal{F}^s$ , its image  $F(\gamma)$  is contained in a leaf of  $\mathcal{F}^s$ .

The main features of the geometric Lorenz flow and its first return map can be seen at figures 1.3 and 1.4.

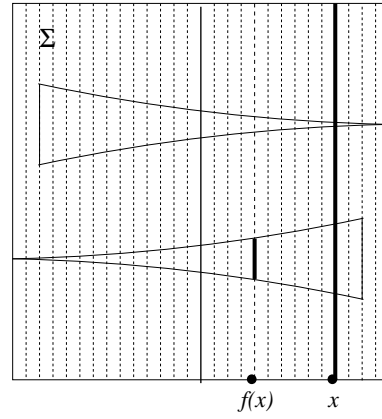
The one-dimensional map  $f$  is obtained quotienting over the leaves of the stable foliation  $\mathcal{F}^s$  defined before.

For a detailed construction of a geometric Lorenz flow see [7, 14].

As mentioned above, a geometric Lorenz attractor is the most representative example of a singular-hyperbolic attractor [23].



**Fig. 1.4** The image  $F(S^*)$ .



**Fig. 1.5** Projection on  $I$ .

### 1.3 The dynamical results

The study of robust attractors is inspired by the Lorenz flow example. Next we list the main dynamical properties of a robust attractor.

#### 1.3.1 Robustness and singular-hyperbolicity

Inspired by the Lorenz flow example we define an equilibrium  $\sigma$  of a flow  $X^t$  to be *Lorenz-like* if the eigenvalues  $\lambda_1, \lambda_2, \lambda_3$  of  $DX(\sigma)$  are real and satisfy the relation at (1.2):

$$\lambda_2 < \lambda_3 < 0 < -\lambda_3 < \lambda_1.$$

These are the equilibria contained in robust attractors naturally, since they are the only kind of equilibria in a 3-flow which cannot be perturbed into saddle-connections which generate sinks or sources when unfolded.

**Theorem 1.** *Let  $\Lambda$  be a robustly transitive set of  $X \in \mathcal{X}^1(M)$ . Then, either for  $Y = X$  or  $Y = -X$ , every singularity  $\sigma \in \Lambda$  is Lorenz-like for  $Y$  and satisfies  $W_Y^{ss}(\sigma) \cap \Lambda = \{\sigma\}$ .*

The fact that a robust attractor does not admit sinks or sources for all nearby vector fields in its isolating neighborhood has several other strong consequences (whose study was pioneered by R. Mañé in its path to solve the Stability Conjecture in [20]) which enable us to show the following, see [25].

**Theorem 2.** *A robustly transitive set for  $X \in \mathcal{X}^1(M)$  is a singular-hyperbolic attractor for  $X$  or for  $-X$ .*

The following shows in particular that the notion of singular hyperbolicity is an extension of the notion of uniform hyperbolicity.

**Theorem 3.** *Let  $\Lambda$  be a singular hyperbolic compact set of  $X \in \mathcal{X}^1(M)$ . Then any invariant compact set  $\Gamma \subset \Lambda$  without singularities is uniformly hyperbolic.*

A consequence of Theorem 3 is that every periodic orbit of a singular hyperbolic set is hyperbolic. The existence of a periodic orbit in every singular-hyperbolic attractor was proved recently in [11] and also a more general result was obtained in [9].

**Theorem 4.** *Every singular hyperbolic attractor  $\Lambda$  has a dense subset of periodic orbits.*

In the same work [9] it was announced that every singular hyperbolic attractor is the homoclinic class associated to one of its periodic orbits. Recall that the *homoclinic class* of a periodic orbit  $\mathcal{O}$  for  $X$  is the closure of the set of transversal intersection points of its stable and unstable manifold:  $H(\mathcal{O}) = \overline{W^u(\mathcal{O}) \cap W^s(\mathcal{O})}$ . This result is well known for the elementary dynamical pieces of uniformly hyperbolic attractors. Moreover, in particular, the geometric Lorenz attractor is a homoclinic class as proved in [10].

### 1.3.2 Singular-hyperbolicity and chaotic behavior

Using the area expansion along the bidimensional central direction, which contains the direction of the flow, one can show

**Theorem 5.** *Every orbit in any singular-hyperbolic attractor has a direction of exponential divergence from nearby orbits (positive Lyapunov exponent).*



Denote by  $S(\mathbb{R})$  the set of surjective increasing continuous real functions  $h : \mathbb{R} \rightarrow \mathbb{R}$  endowed with the  $C^0$  topology. The flow  $X_t$  is *expansive* on an invariant compact set  $\Lambda$  if for every  $\varepsilon > 0$  there is  $\delta > 0$  such that if for some  $h \in S(\mathbb{R})$  and  $x, y \in \Lambda$

$$\text{dist}(X_t(x), X_{h(t)}(y)) \leq \delta \quad \text{for all } t \in \mathbb{R},$$

then  $X_{h(t_0)}(y) \in X_{[t_0-\varepsilon, t_0+\varepsilon]}(x)$ , for some  $t_0 \in \mathbb{R}$ . A stronger notion of expansiveness was introduced by Bowen-Ruelle [12] for uniformly hyperbolic attractors, but equilibria in expansive sets under this strong notion must be isolated, see e.g. [7].

Komuro proved in [16] that a geometrical Lorenz attractor  $\Lambda$  is expansive. In particular, this implies that this kind of attractor is *sensitive with respect to initial data*, i.e., there is  $\delta > 0$  such that for any pair of distinct points  $x, y \in \Lambda$ , if  $\text{dist}(X_t(x), X_t(y)) < \delta$  for all  $t \in \mathbb{R}$ , then  $x$  is in the orbit of  $y$ . In [8] this was fully extended to the singular-hyperbolic setting.

**Theorem 6.** *Let  $\Lambda$  be a singular hyperbolic attractor of  $X \in \mathcal{X}^1(M)$ . Then  $\Lambda$  is expansive.*

**Corollary 1.** *Singular hyperbolic attractors are sensitive with respect to initial data.*

### 1.3.3 Singular-hyperbolicity, positive volume and global hyperbolicity

Recently a generalization of the results of Bowen-Ruelle [12] was obtained in [2] showing that a uniformly hyperbolic transitive subset of saddle-type for a  $C^{1+\alpha}$  flow has zero volume, for any  $\alpha > 0$ . We denote the family of all flows whose differentiability class is at least Hölder- $C^1$  by  $C^{1+}$ .

**Theorem 7.** *A  $C^{1+}$  singular-hyperbolic attractor has zero volume.*

This can be extended to the following dichotomy. Recall that a *transitive Anosov vector field*  $X$  is a vector field without singularities such that the entire manifold  $M$  is a uniformly hyperbolic set of saddle-type.

**Theorem 8.** *Let  $\Lambda$  be a singular hyperbolic attractor for a  $C^{1+}$  3-dimensional vector field  $X$ . Then either  $\Lambda$  has zero volume or  $X$  is a transitive Anosov vector field.*

## 1.4 The Ergodic Theory of singular-hyperbolic attractors

The ergodic theory of singular-hyperbolic attractors is incomplete. Most results still are proved only in the particular case of the geometric Lorenz flow, which automatically extends to the original Lorenz flow after the work of Tucker [31], but demand an extra effort to encompass the full singular-hyperbolic setting.

### 1.4.1 Existence of a physical measure

Another main result obtained in [8] is that typical orbits in the basin of every singular-hyperbolic attractor, for a  $C^2$  flow  $X$  on a 3-manifold, have well-defined statistical behavior, i.e. for Lebesgue almost every point the forward Birkhoff time average converges, and it is given by a certain physical probability measure. It was also obtained that this measure admits absolutely continuous conditional measures along the center-unstable directions on the attractor. As a consequence, it is a  $u$ -Gibbs state and an equilibrium state for the flow.

**Theorem 9.** *A  $C^2$  singular-hyperbolic attractor  $\Lambda$  admits a unique ergodic physical hyperbolic invariant probability measure  $\mu$  whose basin covers Lebesgue almost every point of a full neighborhood of  $\Lambda$ .*

Recall that an invariant probability measure  $\mu$  for a flow  $X$  is *physical* (or *SRB*) if its *basin*

$$B(\mu) = \left\{ x \in M : \lim_{T \rightarrow \infty} \frac{1}{T} \int_0^T \psi(X_t(x)) dt = \int \psi d\mu, \forall \psi \in C^0(M, \mathbb{R}) \right\}$$

has positive volume in  $M$ .

Here hyperbolicity means *non-uniform hyperbolicity* of the probability measure  $\mu$ : the tangent bundle over  $\Lambda$  splits into a sum  $T_z M = E_z^s \oplus E_z^X \oplus F_z$  of three one-dimensional invariant subspaces defined for  $\mu$ -a.e.  $z \in \Lambda$  and depending measurably on the base point  $z$ , where  $\mu$  is the physical measure in the statement of Theorem 9,  $E_z^X$  is the flow direction (with zero Lyapunov exponent) and  $F_z$  is the direction with positive Lyapunov exponent, that is, for every non-zero vector  $v \in F_z$  we have

$$\lim_{t \rightarrow +\infty} \frac{1}{t} \log \|DX_t(z) \cdot v\| > 0.$$

We note that the invariance of the splitting implies that  $E_z^{cu} = E_z^X \oplus F_z$  whenever  $F_z$  is defined.

Theorem 9 is another statement of sensitiveness, this time applying to the whole essentially open set  $B(\Lambda)$ . Indeed, since non-zero Lyapunov exponents express that the orbits of infinitesimally close-by points tend to move apart from each other, this theorem means that most orbits in the basin of attraction separate under forward iteration. See Kifer [17], and Metzger [22], and references therein, for previous results about invariant measures and stochastic stability of the geometric Lorenz models.

The  $u$ -Gibbs property of  $\mu$  is stated as follows.

**Theorem 10.** *Let  $\Lambda$  be a singular-hyperbolic attractor for a  $C^2$  three-dimensional flow. Then the physical measure  $\mu$  supported in  $\Lambda$  has a disintegration into absolutely continuous conditional measures  $\mu_\gamma$  along center-unstable surfaces  $\gamma$  such that  $\frac{d\mu_\gamma}{dm_\gamma}$  is uniformly bounded from above. Moreover  $\text{supp}(\mu) = \Lambda$ .*

Here the existence of unstable manifolds is guaranteed by the hyperbolicity of the physical measure: the strong-unstable manifolds  $W^{uu}(z)$  are the “integral manifolds” in the direction of the one-dimensional sub-bundle  $F$ , tangent to  $F_z$  at almost every  $z \in \Lambda$ . The sets  $W^{uu}(z)$  are embedded sub-manifolds in a neighborhood of  $z$  which, in general, depend only measurably (including its size) on the base point  $z \in \Lambda$ . The *strong-unstable manifold* is defined by

$$W^{uu}(z) = \{y \in M : \lim_{t \rightarrow -\infty} \text{dist}(X_t(y), X_t(z)) = 0\}$$

and exists for almost every  $z \in \Lambda$  with respect to the physical and hyperbolic measure obtained in Theorem 9. We remark that since  $\Lambda$  is an attracting set, then  $W^{uu}(z) \subset \Lambda$  whenever defined. The central unstable surfaces mentioned in the statement of Theorem 10 are just small strong-unstable manifolds carried by the flow, which are tangent to the central-unstable direction  $E^{cu}$ .

The absolute continuity property along the center-unstable sub-bundle given by Theorem 10 ensures that

$$h_\mu(X^1) = \int \log |\det(DX^1 | E^{cu})| d\mu,$$

by the characterization of probability measures satisfying the Entropy Formula, obtained in [19]. The above integral is the sum of the positive Lyapunov exponents along the sub-bundle  $E^{cu}$  by Oseledets Theorem [21, 32]. Since in the direction  $E^{cu}$  there is only one positive Lyapunov exponent along the one-dimensional direction  $F_z$ ,  $\mu$ -a.e.  $z$ , the ergodicity of  $\mu$  then shows that the following is true.

**Corollary 2.** *If  $\Lambda$  is a singular-hyperbolic attractor for a  $C^2$  three-dimensional flow  $X^t$ , then the physical measure  $\mu$  supported in  $\Lambda$  satisfies the Entropy Formula*

$$h_\mu(X^1) = \int \log \|DX^1 | F_z\| d\mu(z).$$

Again by the characterization of measures satisfying the Entropy Formula, we get that  $\mu$  has absolutely continuous disintegration along the strong-unstable direction, along which the Lyapunov exponent is positive, thus  $\mu$  is a *u-Gibbs state* [28]. This also shows that  $\mu$  is an equilibrium state for the potential  $-\log \|DX^1 | F_z\|$  with respect to the diffeomorphism  $X^1$ . We note that the entropy  $h_\mu(X^1)$  of  $X^1$  is the entropy of the flow  $X^t$  with respect to the measure  $\mu$  [32].

Hence we are able to extend most of the basic results on the ergodic theory of hyperbolic attractors to the setting of singular-hyperbolic attractors.

### 1.4.2 Hitting and recurrence time versus local dimension for geometric Lorenz flows

Given  $x \in M$ , let  $B_r(x) = \{y \in M; d(x, y) \leq r\}$  be the ball centered at  $x$  with radius  $r$ . The *local dimension* of  $\mu$  at  $x \in M$  is defined by

$$d_\mu(x) = \lim_{r \rightarrow \infty} \frac{\log \mu(B_r(x))}{\log r}$$

if this limit exists. In this case  $\mu(B_r(x)) \sim r^{d_\mu(x)}$ .

This notion characterizes the local geometric structure of an invariant measure with respect to the metric in the phase space of the system, see [34] and [27].

The existence of the local dimension for a Borel probability measure  $\mu$  on  $M$  implies the crucial fact that virtually *all* the known characteristics of dimension type of the measure coincide. The common value is a fundamental characteristic of the fractal structure of  $\mu$ , see [27].

Let  $x_0 \in \mathbb{R}^3$  and

$$\tau_r^{X^t}(x, x_0) = \inf\{t \geq 0 \mid X^t(x) \in B_r(x_0)\}$$

be the time needed for the  $X$ -orbit of a point  $x$  to enter for the *first time* in a ball  $B_r(x_0)$ . The number  $\tau_r^{X^t}(x, x_0)$  is the *hitting time associated to* the flow  $X^t$  and  $B_r(x_0)$ . If the orbit  $X^t$  starts at  $x_0$  itself and we consider the second entrance time in the ball

$$\tau'_r(x_0) = \inf\{t \in \mathbb{R}^+ : X^t(x_0) \in B_r(x_0), \exists i, s.t. X^i(x_0) \notin B_r(x_0)\}$$

we have a quantitative recurrence indicator, and the number  $\tau'_r(x_0)$  is called the *recurrence time associated to* the flow  $X^t$  and  $B_r(x_0)$ .

Now let  $X^t$  be a geometric Lorenz flow, and  $\mu$  its  $X^t$ -invariant SBR measure.

The main result in [14] establishes the following

**Theorem 11.** *For  $\mu$ -almost every  $x$ ,*

$$\lim_{r \rightarrow 0} \frac{\log \tau_r(\mathbf{x}, \mathbf{x}_0)}{-\log r} = d_\mu(\mathbf{x}_0) - 1.$$

Observe that the result above indicates once more the chaoticity of a Lorenz-like attractor: it shows that asymptotically, such attractors behave as an i.d. system.

We can always define the *upper* and the *lower* local dimension at  $x$  as

$$d_\mu^+(x) = \limsup_{r \rightarrow \infty} \frac{\log \mu(B_r(x))}{\log r}, \quad d_\mu^-(x) = \liminf_{r \rightarrow \infty} \frac{\log \mu(B_r(x))}{\log r}.$$

If  $d^+(x) = d^-(x) = d$  almost everywhere the system is called *exact dimensional*. In this case many properties of dimension of a measure coincide. In particular,  $d$  is equal to the dimension of the measure  $\mu$ :  $d = \inf\{\dim_H Z; \mu(Z) = 1\}$ . This happens

in a large class of systems, for example, in  $C^2$  diffeomorphisms having non zero Lyapunov exponents almost everywhere, [27].

Using a general result proved in [29] it is also proved in [14] a quantitative recurrence bound for the Lorenz geometric flow:

**Theorem 12.** *For a geometric Lorenz flow it holds*

$$\liminf_{r \rightarrow 0} \frac{\log \tau'_r(x)}{-\log r} = d_\mu^- - 1, \quad \limsup_{r \rightarrow 0} \frac{\log \tau'_r(x)}{-\log r} = d_\mu^+ - 1, \quad \mu - a.e..$$

where  $\tau'$  is the recurrence time for the flow, as defined above.

The proof of Theorem 11 is based on the following results, proved in [14].

Let  $F : S \rightarrow S$  be the first return map to  $S$ , a global cross section to  $X^t$  through  $W^s(p)$ ,  $p$  the singularity at the origin, as indicated at Figure 1.3. It follows that  $F$  has a physical measure  $\mu_F$ , see e.g. [33]. Recall that we say the system  $(S, F, \mu_F)$  has exponential decay of correlation for Lipschitz observables if there are constants  $C > 0$  and  $\lambda > 0$ , depending only on the system such that for each  $n$  it holds

$$\left| \int g(F^n(x))f(x)d\mu - \int g(x)d\mu \int f(x)d\mu \right| \leq C \cdot e^{-\lambda n}$$

for any Lipschitz observable  $g$  and  $f$  with bounded variation,

**Theorem 13.** *Let  $\mu_F$  an invariant physical measure for  $F$ . The system  $(S, F, \mu_F)$  has exponential decay of correlations with respect to Lipschitz observables.*

We remark that a sub-exponential bound for the decay of correlation for a two dimensional Lorenz like map was given in [13] and [1].

**Theorem 14.**  $\mu_F$  is exact, that is,  $d_{\mu_F}(x)$  exist almost every  $x \in S$ .

Let  $x_0 \in S$  and  $\tau_r^S(x, x_0)$  be the time needed to  $\mathcal{O}_x$  enter for the first time in  $B_r(x_0) \cap S = B_{r,S}$ .

**Theorem 15.**  $\lim_{r \rightarrow 0} \frac{\log \tau_r(x, x_0)}{-\log r} = \lim_{r \rightarrow 0} \frac{\log \tau_r^S(x, x_0)}{-\log r} = d_{\mu_F}(x_0)$ .

From the fact that the attractor is a suspension of the support of  $\mu_F$  we easily deduce the following.

**Theorem 16.**  $d_\mu(x) = d_{\mu_F}(x) + 1$ .

We remark that the results in this section can be extended to a more general class of flows described in [14]. The interested reader can find the detailed proofs in this article.



The study of suspension (or special) flows is motivated by modeling a flow admitting a cross-section. Such flow is equivalent to a suspension semiflow over the Poincaré return map to the cross-section with roof function given by the return time function on the cross-section. This is a main tool in the ergodic theory of uniformly hyperbolic flows developed by Bowen and Ruelle [12].

### 1.4.3.2 Conditions on the base dynamics

We assume that the singular set  $\mathcal{S}$  (containing the points where  $f$  is either *not defined*, *discontinuous* or *not differentiable*) is regular, e.g. a submanifold of  $M$ , and that  $f$  is *non-uniformly expanding*: there exists  $c > 0$  such that for Lebesgue almost every  $x \in M$

$$\limsup_{n \rightarrow +\infty} \frac{1}{n} S_n \psi(x) \leq -c \quad \text{where} \quad \psi(x) = \log \|Df(x)^{-1}\|.$$

Moreover we assume that  $f$  has *exponentially slow recurrence to the singular set*  $\mathcal{S}$  i.e. for all  $\varepsilon > 0$  there is  $\delta > 0$  s.t.

$$\limsup_{n \rightarrow +\infty} \frac{1}{n} \log \text{Leb} \left\{ x \in M : \frac{1}{n} S_n |\log d_\delta(x, \mathcal{S})| > \varepsilon \right\} < 0,$$

where  $d_\delta(x, y) = \text{dist}(x, y)$  if  $\text{dist}(x, y) < \delta$  and  $d_\delta(x, y) = 1$  otherwise.

These conditions ensure [4] in particular the existence of finitely many ergodic absolutely continuous (in particular *physical*)  $f$ -invariant probability measures  $\mu_1, \dots, \mu_k$  whose basins cover the manifold Lebesgue almost everywhere.

We say that an  $f$ -invariant measure  $\mu$  is an *equilibrium state* with respect to the potential  $\log J$ , where  $J = |\det Df|$ , if  $h_\mu(f) = \mu(\log J)$ , that is if  $\mu$  satisfies the *Entropy Formula*. Denote by  $\mathbb{E}$  the family of all such equilibrium states. It is not difficult to see that each physical measure in our setting belongs to  $\mathbb{E}$ .

We assume that  $\mathbb{E}$  is formed by a unique absolutely continuous probability measure.

### 1.4.3.3 Conditions on the roof function

We assume that  $r : M \setminus \mathcal{S} \rightarrow \mathbb{R}^+$  has *logarithmic growth near  $\mathcal{S}$* : there exists  $K = K(\varphi) > 0$  such that<sup>2</sup>  $r \cdot \chi_{B(\mathcal{S}, \delta)} \leq K \cdot |\log d_\delta(x, \mathcal{S})|$  for all small enough  $\delta > 0$ . We also assume that  $r$  is bounded from below by some  $r_0 > 0$ .

Now we can state the result on large deviations.

**Theorem 17.** *Let  $X^t$  be a suspension semiflow over a non-uniformly expanding transformation  $f$  on the base  $M$ , with roof function  $r$ , satisfying all the previously stated conditions.*

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<sup>2</sup>  $B(\mathcal{S}, \delta)$  is the  $\delta$ -neighborhood of  $\mathcal{S}$ .

Let  $\psi : M_r \rightarrow \mathbb{R}$  be continuous,  $\nu = \mu \times \text{Leb}^1$  be the induced invariant measure<sup>3</sup> for the semiflow  $X^t$  and  $\lambda = \text{Leb} \times \text{Leb}^1$  be the natural extension of volume to the space  $M_r$ . Then

$$\limsup_{T \rightarrow \infty} \frac{1}{T} \log \lambda \left\{ z \in M_r : \left| \frac{1}{T} \int_0^T \psi(X^t(z)) dt - \nu(\psi) \right| > \varepsilon \right\} < 0.$$

#### 1.4.3.4 Consequences for the Lorenz flow

Now consider a Lorenz geometric flow as constructed in Section 1.2 and let  $f$  be the one-dimensional map associated, obtained quotienting over the leaves of the stable foliation, see Figure 1.3. This map has all the properties stated previously for the base transformation. The Poincaré return time gives also a roof function with logarithmic growth near the singularity line.

The uniform contraction along the stable leaves implies that the *time averages of two orbits on the same stable leaf under the first return map are uniformly close* for all big enough iterates. If  $P : S \rightarrow [-1, 1]$  is the projection along stable leaves

**Lemma 1.** *For  $\varphi : U \supset \Lambda \rightarrow \mathbb{R}$  continuous and bounded,  $\varepsilon > 0$  and  $\varphi(x) = \int_0^{r(x)} \psi(x, t) dt$ , there exists  $\zeta : [-1, 1] \setminus \mathcal{S} \rightarrow \mathbb{R}$  with logarithmic growth near  $\mathcal{S}$  such that  $\left\{ \left| \frac{1}{n} S_n^R \varphi - \mu(\varphi) \right| > 2\varepsilon \right\}$  is contained in*

$$P^{-1} \left( \left\{ \left| \frac{1}{n} S_n^f \zeta - \mu(\zeta) \right| > \varepsilon \right\} \cup \left\{ \left| \frac{1}{n} S_n^f \log \text{dist}_\delta(y, \mathcal{S}) \right| > \varepsilon \right\} \right).$$

Hence in this setting it is enough to study the quotient map  $f$  to get information about deviations for the Poincaré return map. Coupled with the main result we are then able to deduce

**Corollary 3.** *Let  $X^t$  be a flow on  $\mathbb{R}^3$  exhibiting a Lorenz or a geometric Lorenz attractor with trapping region  $U$ . Denoting by  $\text{Leb}$  the normalized restriction of the Lebesgue volume measure to  $U$ ,  $\psi : U \rightarrow \mathbb{R}$  a bounded continuous function and  $\mu$  the unique physical measure for the attractor, then for any given  $\varepsilon > 0$*

$$\limsup_{T \rightarrow \infty} \frac{1}{T} \log \text{Leb} \left\{ z \in U : \left| \frac{1}{T} \int_0^T \psi(X^t(z)) dt - \mu(\psi) \right| > \varepsilon \right\} < 0.$$

Moreover for any compact  $K \subset U$  such that  $\mu(K) < 1$  we have

$$\limsup_{T \rightarrow +\infty} \frac{1}{T} \log \text{Leb} \left( \{x \in K : X^t(x) \in K, 0 < t < T\} \right) < 0.$$

---

<sup>3</sup> for any  $A \subset M_r$  set  $\nu(A) = \mu(r)^{-1} \int d\mu(x) \int_0^{r(x)} ds \chi_A(x, s)$ .



### 1.4.3.5 Idea of the proof

We use properties of non-uniformly expanding transformations, especially a large deviation bound recently obtained [5], to deduce a large deviation bound for the suspension semiflow reducing the estimate of the volume of the deviation set to the volume of a certain deviation set for the base transformation.

The initial step of the reduction is as follows. For a continuous and bounded  $\psi : M_r \rightarrow \mathbb{R}$ ,  $T > 0$  and  $z = (x, s)$  with  $x \in M$  and  $0 \leq s < r(x) < \infty$ , there exists the **lap number**  $n = n(x, s, T) \in \mathbb{N}$  such that  $S_n r(x) \leq s + T < S_{n+1} r(x)$ , and we can write

$$\begin{aligned} \int_0^T \psi(X^t(z)) dt &= \int_s^{r(x)} \psi(X^t(x, 0)) dt + \int_0^{T+s-S_n r(x)} \psi(X^t(f^n(x), 0)) dt \\ &\quad + \sum_{j=1}^{n-1} \int_0^{r(f^j(x))} \psi(X^t(f^j(x), 0)) dt. \end{aligned}$$

Setting  $\varphi(x) = \int_0^{r(x)} \psi(x, 0) dt$  we can rewrite the last summation above as  $S_n \varphi(x)$ . We get the following expression for the time average

$$\begin{aligned} \frac{1}{T} \int_0^T \psi(X^t(z)) dt &= \frac{1}{T} S_n \varphi(x) - \frac{1}{T} \int_0^s \psi(X^t(x, 0)) dt \\ &\quad + \frac{1}{T} \int_0^{T+s-S_n r(x)} \psi(X^t(f^n(x), 0)) dt. \end{aligned}$$

Writing  $I = I(x, s, T)$  for the sum of the last two integral terms above, observe that for  $\omega > 0$ ,  $0 \leq s < r(x)$  and  $n = n(x, s, T)$

$$\left\{ (x, s) \in M_r : \left| \frac{1}{T} S_n \varphi(x) + I(x, s, T) - \frac{\mu(\varphi)}{\mu(r)} \right| > \omega \right\}$$

is contained in

$$\left\{ (x, s) \in M_r : \left| \frac{1}{T} S_n \varphi(x) - \frac{\mu(\varphi)}{\mu(r)} \right| > \frac{\omega}{2} \right\} \cup \left\{ (x, s) \in M_r : I(x, s, T) > \frac{\omega}{2} \right\}.$$

The left hand side above is a *deviation set for the observable  $\varphi$  over the base transformation*, while the right hand side will be *bounded by the geometric conditions on  $\mathcal{S}$  and by a deviations bound for the observable  $r$  over the base transformation*.

Analysing each set using the conditions on  $f$  and  $r$  and noting that for  $\mu$ - and Leb-almost every  $x \in M$  and every  $0 \leq s < r(x)$

$$\frac{S_n r(x)}{n} \leq \frac{T+s}{n} \leq \frac{S_{n+1} r(x)}{n} \quad \text{so} \quad \frac{n(x, s, T)}{T} \xrightarrow{T \rightarrow \infty} \frac{1}{\mu(r)},$$

we are able to obtain the asymptotic bound of the Main Theorem.

Full details of the proof are presented in [6].

The interested reader can find the proofs of the results mentioned above in the papers listed below, the references therein, and also in one of IMPA's texts [7] for the XXV Brazilian Mathematical Colloquium.

## Acknowledgments

V.A. was partially supported by FCT-CMUP(Portugal) and CNPq, FAPERJ and PRONEX-Dyn.Systems(Brazil). M.J.P. was partially supported by CNPq, FAPERJ and PRONEX-Dyn.Systems(Brazil). This work was done while M. J. Pacifico was enjoying a post-doc leave at CRM Ennio De Giorgi-Scuola Normale Superiore di Pisa with a CNPq fellowship and also partially supported by CRM Ennio De Giorgi.

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